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**SUBSPACE RECURSIONS AND STRUCTURE ALGORITHMS
FOR SINGULAR SYSTEMS**

NSF GRANT ECS-8518164

ANNUAL PROGRESS REPORT
For 15 April 1986- 2 June 1987

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School of Electrical Engineering
Georgia Institute of Technology
Atlanta, GA 30332

June 7, 1987

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I. SUMMARY OF PROGRESS

References denoted by [] refer to Appendix A, the "Summary of Activities and Research", which provides a list of publications, papers submitted, and other activities. References denoted by {} refer to the References section.

Goals of the Proposal

There were two goals in the proposal. They were:

Develop a complete geometric theory for singular systems based on subspaces which can be computed using recursions in terms of the original plant matrices.

Develop stable algorithms in terms of the original plant matrices to reveal the structure of singular systems.

Progress and Results

There has been significant progress toward both of these goals this year. On a quantitative level, 5 journal papers have been accepted and 3 invited conference papers have appeared. A Special Session on Singular Systems has been proposed for this year's CDC. Five additional journal papers have been submitted.

Several of the ideas mentioned in the proposal have paid off, as have some new ones. The most important results are the development of a connection between a new subspace recursion and a new singular system structure algorithm, the definition of the null-output $(A,E,R(B))$ -invariant subspaces which reveal the possible geometric structure of the closed-loop system, the rigorous definition of the reachability subspaces, and a rigorous comparison of the merits of proportional vs. proportional-plus-derivative feedback. Subsidiary results include the use of Walsh functions to analyze singular systems, a system inversion scheme using Walsh functions, an analysis of large-scale interconnected singular systems, the extension of chained aggregation to singular systems, and some work on the Cayley-Hamilton Theorem and Leverrier's method.

The singular system structure algorithm, chained aggregation, and large-scale system analysis contribute to goal number 2, while the work on subspaces and feedback generally contributes to goal number 1. Both goals are placed into an integrated overall perspective by the recursion/algorithm connection we have developed. Our detailed discussion of results may fittingly begin with this topic.

1. Recursions and Algorithms

We use boldface for subspaces, and for a given matrix M denote by M the range $R(M)$ of M . The nullspace of M is denoted by $N(M)$. Superscript "+" denotes Moore-Penrose inverse, and superscript "-1" denotes inverse image of a linear operator, or the usual inverse, if it exists, of its matrix representation.

Consider the generalized linear dynamical system

$$\begin{aligned} \dot{Ex} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \tag{1}$$

with $x \in R^n$, $u \in R^m$, $y \in R^p$. We assume (1) is regular, that is

$$\Delta(s) \equiv |sE - A| \neq 0. \tag{2}$$

In [5,14] we have defined a new subspace. Define $S \subset R^n$ as an output-nulling (ON) (A,E,B) -invariant subspace for (1) if it satisfies

$$\begin{bmatrix} A \\ C \end{bmatrix} S \subset \begin{bmatrix} E \\ 0 \end{bmatrix} S + \begin{bmatrix} B \\ D \end{bmatrix}. \tag{3}$$

It may be shown that $S \subset R^n$ is an ON (A,E,B) -invariant subspace of (1) if and only if, for any $x(0-) \in S$, there exists an input $u(t)$ such that 1. $y(t) = 0$, 2. $x(t) \in S$ for $t \geq 0$, and 3. $u(t)$ and $x(t)$ have strictly proper rational Laplace transforms.

The family of subspaces satisfying (3) is closed under addition, so that it has a largest member. We symbolize the supremal ON (A,E,B) -invariant subspace as L^* .

The next result shows how to compute L^* .

Theorem 1

Consider the subspace recursion

$$X_{k+1} = \begin{bmatrix} A \\ C \end{bmatrix}^{-1} \left[\begin{bmatrix} E \\ 0 \end{bmatrix} X_k + \begin{bmatrix} B \\ D \end{bmatrix} \right], \text{ with } X_0 = R^n. \tag{4}$$

Then $L^* = X_n$. ■

If $D = 0$, then L^* is equal to V^* , the supremal (A,E,B) -

invariant subspace in $N(C) \{1,2\}$. If, in addition, $C=0$ and $B=0$ then L^* is equal to the initial manifold $H_1 \{3\}$. Both of these subspaces are well known.

One of our main results has been showing how to compute L^* , and hence its special cases V^* and H_1 , using (numerically stable) unitary transformations, as is now described.

Consider the following Singular System Structure Algorithm performed on the regular system (1):

Algorithm 2: (Structure Algorithm)

0. Initialize:

Set $k=0$.

Define $E_0 = E$, $A_0 = A$, $B_0 = B$, $C_0 = C$, $D_0 = D$, $\underline{C}_0 = 0$.

1. Iteration k:

Find constant unitary transformations T_k and S_k such that

$$T_k \begin{bmatrix} \overset{n}{E_k} & \overset{n}{A_k} & \overset{m}{B_k} \\ \underline{\overset{n}{C_k}} & 0 & 0 \end{bmatrix} \begin{bmatrix} r_k \\ t_k \end{bmatrix} = \begin{bmatrix} \overset{n}{E_{k+1}} & \overset{n}{A_{k+1}} & \overset{m}{B_{k+1}} \\ 0 & \underline{\overset{n}{A_k}} & \underline{\overset{m}{B_k}} \end{bmatrix} \begin{bmatrix} r_{k+1} \\ t_{k+1} \end{bmatrix} \quad (5a)$$

$$S_k \begin{bmatrix} \overset{n}{C_k} & \overset{m}{D_k} \\ \underline{\overset{n}{A_k}} & \underline{\overset{m}{B_k}} \end{bmatrix} s_k = \begin{bmatrix} \overset{n}{C_{k+1}} & \overset{m}{D_{k+1}} \\ \underline{\overset{n}{C_{k+1}}} & 0 \end{bmatrix} \begin{bmatrix} s_{k+1} \\ t_{k+1} \end{bmatrix} \quad (5b)$$

with E_{k+1} and D_{k+1} having full row rank r_{k+1} and s_{k+1} respectively.

If $t_{k+1} = 0$ or $\underline{C}_{k+1} = 0$ go to 2. Otherwise set $k = k+1$ and go to step 1.

2. Define $L = k+1$. End.

This algorithm was used in studying large-scale interconnections of singular systems in [4]. Note that if $E = I$ it is Silverman's structure algorithm [4], while if $B = 0$, $C = 0$, $D = 0$ it reduces to Luenberger's shuffle algorithm [5].

We can now relate Algorithm (5) to recursion (4).

Theorem 3

Perform recursion (4) and Algorithm (5) on the regular system (1). Then

$$X_k = N \begin{bmatrix} \underline{C}_0 \\ \underline{C}_1 \\ \vdots \\ \underline{C}_k \end{bmatrix} = \bigcap_{i=0}^k N(\underline{C}_i) . \quad (6)$$

(Note: if $i > L$, then interpret $\underline{C}_i = 0$.)

This theorem generalizes a well known state-space result {4}. Algorithm (5) can also be related to the inversion problem, since (1) is left invertible if and only if $\text{rank}(E_L) = n$ and $\text{rank}(D_L) = m$ [14].

2. Geometric Structure and Feedback

Denote the finite spectrum of (E, A) as $\sigma(E, A)$ and the spectrum of a single matrix F as $\sigma(F)$. Let S provide a basis for S so that (3) implies

$$AS = ESF - BG \quad (7a)$$

$$CS = -DG \quad (7b)$$

for some F and G , with F chosen square, or

$$s \begin{bmatrix} E \\ 0 \end{bmatrix} S - \begin{bmatrix} A \\ C \end{bmatrix} S = s \begin{bmatrix} E \\ 0 \end{bmatrix} S - \begin{bmatrix} E \\ 0 \end{bmatrix} SF + \begin{bmatrix} B \\ D \end{bmatrix} G \quad (8)$$

where s is a complex variable. This can be written as

$$\begin{aligned} (sE - A)S(sI - F)^{-1} &= ES + BG(sI - F)^{-1} \\ 0 &= CS(sI - F)^{-1} + DG(sI - F)^{-1} . \end{aligned} \quad (9)$$

Defining a feedback

$$u = Kx \quad (10)$$

with

$$K = GS^+, \quad (11)$$

(9) becomes

$$(sE - (A+BK)) S(sI-F)^{-1} = ES \quad (12)$$

$$(C+DK) S(sI-F)^{-1} = 0 \quad (13)$$

We now see the important fact that S is a closed-loop (A,E) -invariant subspace on which there has been assigned the spectrum $\sigma(F)$.

In [14] we show the connection between the observability of (1) and the ability to assign a desired spectrum on S , and the connection between the controllability of (1) and spectrum assignability with closed-loop regularity. We show how to use the generalized Lyapunov equation (7) to compute the required feedback. In [7,8] this equation was investigated in connection with the analysis of (1) using Walsh functions. It was shown that (7a) has a solution if and only if

$$\sigma(E,A) \cap \sigma(F) = \emptyset \quad (14)$$

the empty set.

3. Geometric Theory

We have laid a firm foundation for the geometric theory of singular systems [6,9,11,12,14], which we outline here.

Consider the two recursions

$$X_{k+1} = K \cap A^{-1}(EX_k + B), \quad X_0 = K \quad (15)$$

$$Y_{k+1} = K \cap E^{-1}(AY_k + B), \quad Y_0 = 0 \quad (16)$$

with $K \subset R^n$ a given subspace. The first of these is just (4) with $D = 0$, $N(C) = K$, so that structure algorithm (5) may be used to implement it. Likewise, (16) may be implemented with (5) if E and A are interchanged.

Define $V^* = X_\infty$ in (15) with $K = R^n$ and $S_* = Y_\infty$ in (16) with $K = R^n$. Then V^* is the supremal (A,E,B) -invariant subspace of (1) $\{1,2\}$ while S_* is the infimal (E,A,B) -invariant subspace of (1) [9,11,12]. The next results summarize the work in [9,11,12].

The reachability pencil for (1) is

$$R(s) = [sE - A \quad B] \quad (17)$$

The system is said to be controllable if $R(s)$ has no finite or infinite zeros, and reachable if $R(s)$ has no finite zeros and

$$\text{rank}[E \ B] = n. \quad (18)$$

The reachable subspace R is the set of all states reachable from $x(0) = 0$, and the controllable subspace C is the set of all $x(0)$ such that $x(T) = 0$ for some $u(t)$ and time $T > 0$. From previous work [2] it is known that

$$R = V^* \cap S_* \quad (19)$$

$$C = (V^* + N(E)) \cap S_*. \quad (20)$$

In [9], we show how to select a feedback $u = Fx$ using a minimal polynomial basis for the right nullspace of $R(s)$ that assigns the eigenvalues of the closed-loop system

$$Ex = (A + BF)x \quad (21)$$

while guaranteeing closed-loop regularity. Known results [6,7] were extended by showing the relation between the possible closed-loop structure and the column-minimal indices of $R(s)$, which we defined as the controllability indices of (1).

It was shown that the controllability indices may be generated from the dimensions of the subspaces X_* and Y_* in (15) and (16).

The extension of the notion of the reachability subspace to singular systems is not easy. Consider the feedback

$$u = Fx + Gv \quad (22)$$

and define $A_F = (A + BF)$, $B_F = BG$. If there exist F and G such that $(sE - A_F)$ is regular and

$$S = R[B_F \ A_FB_F \ \dots \ (A_F)^{n-1}B_F] \quad (23)$$

then S is called a reachability subspace for (1). The reachable subspace R is the largest reachability subspace.

In [12] we showed the main result that S is a reachability subspace if and only if

$$AS \subset ES + B, \quad (24)$$

that is, S is an (A, E, B) -invariant subspace, and S is the limit of (16) with $K = S$. This direct generalization of Wonham's result [8] is attained only after considerable work once the correct notion of a "friend" of S has been defined. A straightforward extension of Wonham's notion results in a "bad friend" which will certainly result in closed-loop irregularity unless stringent and unreasonable conditions hold.

In [12] we considered conditions under which (1) may be regularized by feedback (22) in the event that (2) does not hold. This work results in an extension of the notions of reachability and controllability to definitions which are feedback invariant. In the previous literature, the curious situation had obtained that, although it was desired for the controllability properties of (1) to be invariant under feedback of the form (22), the definitions themselves were not invariant, since (22) can destroy regularity in (1). This situation was corrected in [12].

In the context of this discussion, it was natural to consider the relation between proportional (P) feedback (22) and proportional-plus-derivative (PD) feedback

$$u = Kx + F\dot{x} + Gv. \quad (25)$$

It was discovered that the latter offers virtually no advantage over the former. This is an unfortunate state of affairs, because if PD feedback is allowed, the geometric notions extend in a far easier manner [9]. Our results show clearly the need for a geometric theory based on P feedback, since it is easier to implement than PD feedback.

The essential result of [12] is that (1) is regularizable by P or PD feedback (the condition is the same) if and only if

$$EV^* + A + B = R^n. \quad (26)$$

It is interesting to compare this to the well known condition for controllability at infinity [10,11]

$$E + AN(E) + B = R^n. \quad (27)$$

4. Miscellaneous Results

Several other results and connections have been derived. They are mentioned on p. 2, and lack of space precludes their discussion here. Under separate cover, copies of this year's publications will be forwarded to NSF as what may be considered Appendix B to this report.

II. CURRENT DEVELOPMENTS AND SUMMARY OF WORK FOR THE SECOND YEAR

In point of fact, the Proposed Goals on page 2 have for the most part already been achieved. However, a number of important developments are on the horizon. Some of them are briefly discussed here.

1. System Inversion for Singular Systems

A general system inversion algorithm for (1) can be developed using a variant of the structure algorithm (5). This is not a trivial problem, but the results in [4] can be extended in a neat fashion into an inversion algorithm. The details are being worked out. This tack will also result in an extension of the geometric tests for inversion of state-variable systems that can be formulated in terms of the subspace L^* introduced on p. 3 (c.f. {8}).

2. Chained Aggregation and Hessenberg Forms

Two Hessenberg forms for (sE-A) have been used by Van Dooren {12} and by K. Clark {13}. We are pursuing a point of view which relates them both to Chained Aggregation for singular systems. A quotient space theory has been developed for the pencil (sE-A) so that this work may be interpreted geometrically [13], and connections with the QZ algorithm are being looked at.

The use of the Hessenberg forms in solving design problems for (1) is also being examined. These forms should be useful in eigenstructure assignment, solution of the generalized Riccati equations, and so on.

3. Output-Nulling Subspaces and Riccati Equations

The output-nulling subspaces (3) may be intersected with their duals to define the output-nulling reachability subspaces. Results like those of Molinari {14}, which show the connection of these subspaces with the Riccati equations, should then extend to the singular case. Algorithm (5) or a variant of it should be useful for solving the singular Riccati equations.

4. Variable-Structure Systems

A variable-structure system (VSS) on the sliding mode is nothing but a singular system {15}. Thus, designing the sliding mode for desired performance is just the problem of assigning the

closed-loop admissible manifold of (1). This connection will be pursued to study variable-structure systems when the usual assumptions do not apply so that current VSS theory may not be used.

5. Stochastic Singular Systems

It is known from the work of Malhami {16} that stochastic state systems with hysteresis obey Fokker-Planck equations that govern the propagation of two probability density functions in a fashion similar to the wave packet description of a particle in a square well in quantum mechanics. R. Newcomb {17} has shown that a continuous state-space system with hysteresis may be represented as a singular system (1). Therefore, the theory in {16} should extend to at least a special class of singular systems. We plan to investigate further during the upcoming year.

III. CONCLUSION

Our contention is that both of the Proposed Goals on p. 2 have for the most part been achieved. However, much work remains to be done to pursue some exciting ideas on the horizon. The study of singular systems turns out to produce surprising results that at first glance seem paradoxical, but then on further examination lend surprising insight that even illuminates what is happening in the state-space case {18}.

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in linear multivariable control," IEEE Trans. Automat. Control, pp. 761-764, Oct. 1976.

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17. R.W. Newcomb, "Semistate design theory: binary and swept hysteresis," J. Circuits, Systems, Signal Proc., Vol. 1, No. 2, pp. 203-216, 1982.

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APPENDIX A
SUMMARY OF ACTIVITIES AND RESEARCH

I. Papers Published

1. F.L. Lewis, "A survey of linear singular systems," J. Circuits, Systems, Signal Proc., special issue on Semistate Systems, Vol. 5, No. 1, pp. 3-36, 1986.
2. F.L. Lewis, B.L. Stevens, R.S. Kemp, and R.D. Marshall, "Eigenstructure assignment by dynamic output feedback," Proc. ACC, pp. 963-969, June 1986.
3. F.L. Lewis, "Further remarks on the Cayley-Hamilton theorem and Leverrier's method for the matrix pencil (sE-A)," IEEE Trans. Automat. Control, Vol. AC-31, No. 9, pp. 869-870, Sept. 1986.
4. F.L. Lewis and K. Özçaldıran, "Large-scale dynamical interconnections of singular systems," Proc. MTNS, June 1987.
5. F.L. Lewis and G. Beauchamp, "Computation of subspaces for singular systems," Proc. MTNS, June 1987.
6. K. Özçaldıran and F.L. Lewis, "A brief survey of geometric theory for singular systems," First Int. Conference on Ind. and Appl. Math., Paris, June 1987.
7. F.L. Lewis and B.G. Mertzios, "Analysis of singular systems using orthogonal functions," IEEE Trans. Automat. Control, Vol. AC-32, No. 6, pp. 527-530, June 1987.
8. F.L. Lewis, M.A. Christodoulou, and B.G. Mertzios, "System inversion using orthogonal functions," J. Circuits, Systems, Signal Proc., to appear, 1987.
9. K. Özçaldıran and F.L. Lewis, "A geometric approach to eigenstructure assignment for singular systems," IEEE Trans. Automat. Control, to appear, 1987.

II. Papers Submitted

10. B.G. Mertzios, F.L. Lewis, and G. Vachtsevanos, "On the analysis of singular systems using orthogonal functions," Proc. IEE, submitted, 1986.
11. K. Özçaldıran and F.L. Lewis, "Generalized reachability subspaces for singular systems," SIAM J. Control and Opt., submitted, 1987.
12. K. Özçaldıran and F.L. Lewis, "On the regularizability of

singular systems," IEEE Trans. Automat. Control, submitted, 1987.

13. F.L. Lewis, M.A. Christodoulou, B.G. Mertzios, and K. Özğaldiran, "Chained aggregation of singular systems," submitted, 1987.

14. F.L. Lewis and K. Özğaldiran, "Geometric structure and feedback in singular systems," submitted, 1987.

III. Other Activities

1. Sponsored a Visiting Professorship for K. Özğaldiran, Bosporus University, Istambul, Turkey during Sept.-Dec. 1986. This resulted in several papers.

2. Established a Research Group in Singular Systems with B.G. Mertzios, University of Xanthi, Greece, and M.A. Christodoulou, Patras University, Greece.

3. Proposed a "Special Session on Singular Systems" for the 1987 CDC.

NATIONAL SCIENCE FOUNDATION
Washington, D.C. 20550

FINAL PROJECT REPORT
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PART I—PROJECT IDENTIFICATION INFORMATION

1. Institution and Address School of Elect. Eng. Georgia Inst. of Technology Atlanta, GA 30332	2. NSF Program STOR, Division of ECSE	3. NSF Award Number ECS-8518164
	4. Award Period From 4/86 To 9/88	5. Cumulative Award Amount \$54,134

6. Project Title
Subspace Recursions and Structure Algorithms For Singular Systems

PART II—SUMMARY OF COMPLETED PROJECT (FOR PUBLIC USE)

The goals of the proposal were to develop a complete geometric theory for singular systems and to develop stable algorithms to reveal the structure of these systems. Singular systems are those described by differential equations of the form $\dot{E}x = Ax + Bu$, with E a singular matrix.

During the grant research, the notions of controlled and conditioned invariant subspaces were extended to singular systems. Recursions and stable numerical algorithms were derived to compute these subspaces, and their importance was shown in the semistate feedback and observer design problems respectively.

The reachability subspaces were also rigorously defined for singular systems. This allowed a comparison between the relative merits of proportional and proportional-plus-derivative semistate feedback. Conditions were derived for the existence of a well-defined closed-loop system in both cases, and surprisingly enough, it was shown that the advantages of including a derivative semistate feedback term are small. A geometric analysis of closed-loop pole placement was also developed.

The idea of chained aggregation was extended to singular systems. This involved proving some results on quotient spaces and induced maps for the case when two operators (E and A) act simultaneously on a space.

A Walsh function analysis was carried out for singular and bilinear systems, yielding various generalized Sylvester equations whose solution properties were analyzed.

A singular system structure algorithm was studied and applied to large-scale interconnected systems. Recursive algorithms were developed to solve the singular equations in the three cases of prescribed initial, final, and split boundary conditions.

PART III—TECHNICAL INFORMATION (FOR PROGRAM MANAGEMENT USES)

1. ITEM (Check appropriate blocks)	NONE	ATTACHED	PREVIOUSLY FURNISHED	TO BE FURNISHED SEPARATELY TO PROGRAM	
				Check (✓)	Approx. Date
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b. Publication Citations		✓			
c. Data on Scientific Collaborators	✓				
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e. Technical Description of Project and Results		✓			
f. Other (specify) copies of papers published			✓		
2. Principal Investigator/Project Director Name (Typed) F. L. Lewis	3. Principal Investigator/Project Director Signature			4. Date 30 Jan. 89	

**SUBSPACE RECURSIONS AND STRUCTURE ALGORITHMS
FOR SINGULAR SYSTEMS**

NSF GRANT ECS-8518164
April 1986 - Sept. 1988

FINAL REPORT

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Georgia Institute of Technology
Atlanta, GA 30332
404-894-2994

January 28, 1989

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I. INTRODUCTION

There were two goals in the proposal. They were to:

Develop a complete geometric theory for singular systems based on subspaces which can be computed using recursions in terms of the original plant matrices.

Develop stable algorithms in terms of the original plant matrices to reveal the structure of singular systems.

A summary of activities and research under the grant appears in Appendix A. On a quantitative level, 1 conference Proceedings was published, 17 journal papers were accepted, and 7 invited conference papers appeared. An International Symposium on Singular systems was hosted in Atlanta during December 1987 and 3 invited conference sessions were organized. A special issue of Circuits, Systems, and Signal Processing on singular systems will shortly appear.

Several of the ideas mentioned in the proposal have paid off, as have some new ones. The most important results are:

1. the development of a connection between a new subspace recursion and a singular system structure algorithm,
2. the definition of the null-output $(A, E, R(B))$ -invariant subspaces which reveal the possible geometric structure of the closed-loop system under feedback,
3. the definition of the unknown-input $(N(C), E, A)$ -conditioned invariant subspaces which reveal the geometric structure of the observer problem,
4. the rigorous definition of the reachability subspaces,
5. a rigorous comparison of the merits of proportional vs. proportional-plus-derivative feedback,
6. the use of Walsh functions to analyze singular systems, bilinear systems, and singular 2-D systems,
7. a system inversion scheme using Walsh functions,
8. development of analytical results for 2-D singular systems, including solution techniques and algorithms for computing the 2-D transfer function and fundamental matrix sequence,
9. an analysis of large-scale interconnected singular systems,
10. the extension of chained aggregation to singular systems,

- including a quotient space result involving deflating subspaces,
11. work on the Cayley-Hamilton Theorem and Leverrier's method,
 12. work on discrete singular systems which included a study of the fundamental matrix sequence and recursive solution techniques for the symmetric boundary-value problem,
 13. an approach to analysis and design in singular, bilinear, and 2-D singular systems which relies on the solutions to various Lyapunov equations.

Since copies of papers have been sent to NSF as they were prepared, only the highlights of the research will appear in this report.

II. CONTROLLED INVARIANCE AND FEEDBACK

We use boldface for subspaces, and for a given matrix M denote by \mathbf{M} the range $R(M)$ of M . The nullspace of M is denoted by $N(M)$. Superscript "+" denotes Moore-Penrose inverse, and superscript "-1" denotes inverse image of a linear operator, or the usual inverse, if it exists, of its matrix representation. The references appear in Appendix A.

Consider the generalized linear dynamical system

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} &= \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} \end{aligned} \tag{2.1}$$

with $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{u} \in \mathbb{R}^m$, $\mathbf{y} \in \mathbb{R}^p$. We assume (2.1) is regular, that is

$$\Delta(s) \equiv |s\mathbf{E} - \mathbf{A}| \neq 0. \tag{2.2}$$

We have defined $\mathbf{S} \subset \mathbb{R}^n$ as an output-nulling (ON) (A,E,B)-invariant subspace for (2.1) if it satisfies

$$\begin{bmatrix} \mathbf{A} \\ \mathbf{C} \end{bmatrix} \mathbf{S} \subset \begin{bmatrix} \mathbf{E} \\ \mathbf{0} \end{bmatrix} \mathbf{S} + \begin{bmatrix} \mathbf{B} \\ \mathbf{D} \end{bmatrix}. \tag{2.3}$$

It may be shown that $\mathbf{S} \subset \mathbb{R}^n$ is an ON (A,E,B)-invariant subspace of (2.1) if and only if, for any $\mathbf{x}(0) \in \mathbf{S}$, there exists an input $\mathbf{u}(t)$ such that 1. $\mathbf{y}(t) = \mathbf{0}$, 2. $\mathbf{x}(t) \in \mathbf{S}$ for $t \geq 0$, and 3. $\mathbf{u}(t)$ and $\mathbf{x}(t)$ have strictly proper rational Laplace transforms.

Much of our approach relies on the generalized Lyapunov (or Sylvester) equation

$$\begin{bmatrix} A \\ C \end{bmatrix} S = \begin{bmatrix} E \\ 0 \end{bmatrix} SF - \begin{bmatrix} B \\ D \end{bmatrix} G. \quad (2.4)$$

If S is a basis for \mathcal{S} , so that $\mathcal{S} = \mathcal{R}(S)$, then containment (2.3) is equivalent to (2.4) holding for some F and G .

The family of subspaces satisfying (2.3) is closed under addition, so that it has a largest member. We symbolize the supremal (A, E, B) -invariant subspace as L^* .

The next result shows how to compute L^* .

Theorem 2.1

Consider the subspace recursion

$$\mathbf{x}_{k+1} = \begin{bmatrix} A \\ C \end{bmatrix}^{-1} \left[\begin{bmatrix} E \\ 0 \end{bmatrix} \mathbf{x}_k + \begin{bmatrix} B \\ D \end{bmatrix} \right], \text{ with } \mathbf{x}_0 = \mathbf{r}^n. \quad (2.5)$$

Then $L^* = \mathbf{x}_n$.

■

If $D = 0$, then L^* is equal to \mathbf{v}^* , the supremal (A, E, B) -invariant subspace in $\mathcal{N}(C)$. If, in addition, $C = 0$ and $B = 0$ then L^* is equal to the initial manifold \mathbf{H}_I . Both of these subspaces are well-known.

One of our main results has been showing how to compute L^* , and hence its special cases \mathbf{v}^* and \mathbf{H}_I , using (numerically stable) unitary transformations, as is now described.

Consider the following Singular System Structure Algorithm performed on the regular system (2.1):

Algorithm 2.2: (Structure Algorithm)

0. Initialize:

Set $k = 0$.

Define $E_0 = E$, $A_0 = A$, $B_0 = B$, $C_0 = C$, $D_0 = D$, $\underline{C}_0 = 0$, $W_0 = 0$.

1. Iteration k :

Find constant unitary transformations T_k and S_k such that

$$T_k \begin{bmatrix} \overset{n}{E_k} & \overset{n}{A_k} & \overset{m}{B_k} \\ \underline{\overset{n}{C_k}} & 0 & 0 \end{bmatrix} \begin{bmatrix} r_k \\ t_k \end{bmatrix} = \begin{bmatrix} \overset{n}{E_{k+1}} & \overset{n}{A_{k+1}} & \overset{m}{B_{k+1}} \\ 0 & \underline{\overset{n}{A_k}} & \underline{\overset{m}{B_k}} \end{bmatrix} \begin{bmatrix} r_{k+1} \\ t_{k+1} \end{bmatrix} \quad (2.6a)$$

$$S_k \begin{bmatrix} \overset{n}{C_k} & \overset{m}{D_k} \\ \underline{\overset{n}{A_k}} & \underline{\overset{m}{B_k}} \end{bmatrix} \begin{bmatrix} s_k \\ t_k \end{bmatrix} = \begin{bmatrix} \overset{n}{C_{k+1}} & \overset{m}{D_{k+1}} \\ \underline{\overset{n}{C_{k+1}}} & 0 \end{bmatrix} \begin{bmatrix} s_{k+1} \\ t_{k+1} \end{bmatrix} \quad (2.6b)$$

with E_{k+1} and D_{k+1} having full row rank r_{k+1} and s_{k+1} respectively.

Define

$$W_{k+1} = \begin{bmatrix} W_k \\ \underline{C_{k+1}} \end{bmatrix} . \quad (2.6c)$$

If $\text{rank}(W_{k+1}) = \text{rank}(W_k)$ go to 2. Otherwise set $k = k+1$ and go to step 1.

2. Define $L = k+1$. End.

■

This algorithm was used in studying large-scale interconnections of singular systems in [19]. Note that if $E = I$ it is Silverman's structure algorithm, while if $B = 0$, $C = 0$, $D = 0$ it reduces to Luenberger's shuffle algorithm.

We can now relate Algorithm 2.2 to recursion (2.5).

Theorem 2.3

Perform recursion (2.5) and Algorithm 2.2 on the regular system (2.1). Then, for $0 \leq k \leq L$,

$$x_k = N(W_k) = \bigcap_{i=0}^k N(\underline{C_i}) . \quad (2.7)$$

■

This theorem generalizes a well-known state-space result. Algorithm 2.2 can also be related to the inversion problem, since (2.1) is left invertible if and only if $\text{rank}(E_L) = n$ and $\text{rank}(D_L) = m$ [20].

The usefulness of these open-loop notions in describing what may be achieved by semistate feedback is next discussed.

Denote the finite spectrum of (E,A) as $\sigma(E,A)$ and the spectrum of a single matrix F as $\sigma(F)$. Note that the Lyapunov equation (2.4) is equivalent to

$$s \begin{bmatrix} E \\ 0 \end{bmatrix} S - \begin{bmatrix} A \\ C \end{bmatrix} S = s \begin{bmatrix} E \\ 0 \end{bmatrix} S - \begin{bmatrix} E \\ 0 \end{bmatrix} S F + \begin{bmatrix} B \\ D \end{bmatrix} G \quad (2.8)$$

where s is a complex variable. This can be written as

$$\begin{aligned} (sE-A)S(sI-F)^{-1} &= ES + BG(sI-F)^{-1} \\ 0 &= CS(sI-F)^{-1} + DG(sI-F)^{-1} . \end{aligned} \quad (2.9)$$

Defining a feedback

$$u = Kx \quad (2.10)$$

with K any solution to

$$KS = G, \quad (2.11)$$

(2.9) becomes

$$(sE - (A+BK)) S(sI-F)^{-1} = ES \quad (2.12)$$

$$(C+DK) S(sI-F)^{-1} = 0 . \quad (2.13)$$

Comparing these equations to the Laplace transform of (2.1), we now see the important fact that S is a closed-loop (A,E) -invariant subspace on which there has been assigned the spectrum $\sigma(F)$.

In [12] we show the connection between the observability of (2.1) and the ability to assign a desired spectrum on S , and the connection between the controllability of (2.1) and spectrum assignability with closed-loop regularity. We show how to use the generalized Lyapunov equation (2.4) to compute the required feedback. It was shown that the closed-loop system with feedback given by (2.11) is regular on S if and only if

$$N(E) \cap S = 0. \quad (2.14)$$

III. CONDITIONED INVARIANCE AND OBSERVERS

Substituting $(E^T, A^T, C^T, B^T, D^T)$ for (E, A, B, C, D) , setting $\bar{T} = S^\perp$, and using standard duality results, the dual of (2.3) is found to

be

$$[A \ B] \left[\begin{bmatrix} E^{-1} \underline{T} \\ R^a \underline{T} \end{bmatrix} \cap N[C \ D] \right] \subset \underline{T}. \quad (3.1)$$

Similarly, by dualizing (2.4), it is straightforward to demonstrate that (3.1) is equivalent to the Lyapunov equation

$$T[A \ B] = FT[E \ 0] - G[C \ D] \quad (3.2)$$

having a solution T such that $\underline{T} = N(T)$, for some F and G , with T^T a basis (i.e. T of full row rank).

The containment (3.1) has two unfortunate drawbacks. First, its interpretation is not as straightforward as is that of (2.3). The second drawback is a consequence of the presence of E in (2.1). In the state-space case, similarity of matrix pencils is defined by $U(sI-A)U^{-1}$ for nonsingular U , so that the domain and codomain spaces are the same. However, in the case of the generalized pencil, equivalence is defined as $U(sE-A)V$ for nonsingular U and V . Thus, different transformations are applied in the domain and codomain. The result is that the domain and codomain spaces cannot be considered as the same space.

While \underline{S} is in the domain, inspection of (3.1) reveals that \underline{T} , the formal dual of \underline{S} , is in the codomain. Therefore, we shall define \underline{T} as an unknown-input (UI) $(N(C), E, A)$ -conditioned invariant subspace if

$$\underline{T} = E^{-1} \underline{T}, \quad (3.3)$$

where \underline{T} satisfies (3.1). Then, \underline{T} is a subspace of the domain. We use the underbar to denote subspaces of the codomain.

Note that \underline{T} is an UI $(N(C), E, A)$ -conditioned invariant subspace if and only if

$$\underline{T} = E^{-1} N(T) = N(TE), \quad (3.4)$$

with T a full-row-rank solution to (3.2) for some F and G .

The family of subspaces satisfying (3.1) is closed under intersection, so that it has a unique infimal member \underline{T}_* . The simplest proof of this is to use duality on (2.3). A formal dualization of recursion (2.5) gives the next result.

Theorem 3.1

Define a subspace sequence by

$$\underline{T}_{k+1} = [A \ B] \left[\begin{bmatrix} E^{-1} \underline{T}_k \\ R^m \end{bmatrix} \cap N[C \ D] \right], \text{ with } \underline{T}_0 = 0. \quad (3.5)$$

Then $\underline{T}_\infty = \underline{T}_\infty$. ■

Note that, for each value of k , \underline{S}_k given by (2.5) and \underline{T}_k given by (3.5) are duals.

Subspace \underline{T}_∞ may also be computed using the dual of Algorithm 2.2.

The next results from [17] show that the conditioned invariants are important in the output-injection, or observer design, problem.

Theorem 3.2

\underline{T} satisfies (3.1), or equivalently $\underline{T} = N(T)$, with T a full-row-rank solution to (3.2) for some F and G , if and only if there exists an output injection matrix L such that

$$[A+LC \ B+LD] \begin{bmatrix} E^{-1} \underline{T} \\ R^m \end{bmatrix} \subset \underline{T}. \quad (3.6)$$

■

Applying an output-injection L to (2.1) yields

$$\dot{E}x = (A+LC)x + (B+LD)u. \quad (3.7)$$

Therefore, by comparing (3.6) to (3.1), we see that any UI $(N(C), E, A)$ -conditioned invariant subspace $\underline{T} = E^{-1} \underline{T}$ can be assigned as a closed-loop UI (E, A) -invariant subspace using output injection.

The connection of the UI conditioned invariants with observer design can now be made.

Theorem 3.3

Let \underline{T} satisfy (3.1), so that $\underline{T} = N(T)$, with T a full-row-rank solution to (3.2) for some F and G . Suppose F is asymptotically stable. Then there exists a regular observer with spectrum $\sigma(F)$ which reconstructs $x(t)$ modulo $\underline{T} = E^{-1} \underline{T}$ without knowledge of the input $u(t)$. ■

This result justifies our characterization of subspaces satisfying (3.1) as "unknown-input" subspaces.

We call an observer that is driven only by the output, and not the input, a one-degree-of-freedom observer. The next result shows the best we can do in observing the semistate $x(t)$ of (2.1) using such an observer.

Theorem 3.4

Let $T_* \equiv E^{-1}T_*$ be an infimal subspace satisfying (3.1), (3.3). Then, using a one-degree-of-freedom observer:

a If input measurements are allowed, $x(t)$ can be reconstructed modulo R_* , where

$$R_* \equiv N(C) \cap T_* = N(C) \cap E^{-1}T_*. \quad (3.8)$$

b. If $D \neq 0$, the best that can be achieved without knowledge of the input $u(t)$ is to reconstruct $x(t)$ modulo T_* .

c. If $D = 0$, the best that can be achieved without knowledge of the input $u(t)$ is to reconstruct $x(t)$ modulo R_* . ■

According to this theorem, if input measurements are allowed, or if $D = 0$ and input measurements are not allowed, then the entire semistate may be reconstructed using a one-degree-of-freedom observer if and only if $R_* = 0$.

Note that if $D = 0$, R_* is identical to the supremal almost reachability subspace contained in $N(C)$.

IV. GEOMETRIC THEORY

We have laid a firm foundation for the geometric theory of singular systems [5,9,10,12,21], which we outline here.

Consider the two recursions

$$X_{k+1} = K \cap A^{-1}(EX_k + B), \quad X_0 = K \quad (4.1)$$

$$Y_{k+1} = K \cap E^{-1}(AY_k + B), \quad Y_0 = 0 \quad (4.2)$$

with $K \subset R^n$ a given subspace. The first of these is just (2.5) with $D = 0$, $N(C) = K$, so that structure Algorithm 2.2 may be used to implement it. Likewise, (4.2) may be implemented with Algorithm 2.2 if E and A are interchanged.

Define $V^* = X_n$ in (4.1) with $K = R^n$ and $S_* = Y_n$ in (4.2) with $K = R^n$. Then V^* is the supremal (A, E, B) -invariant subspace of (2.1) while S_* is the infimal (E, A, B) -invariant subspace of (2.1). The next results summarize the work in the above mentioned references.

The reachability pencil for (2.1) is

$$R(s) = [sE - A \quad B] . \quad (4.3)$$

The system is said to be controllable if $R(s)$ has no finite or infinite zeros, and reachable if $R(s)$ has no finite zeros and

$$\text{rank}[E \quad B] = n . \quad (4.4)$$

The reachable subspace \mathcal{R} is the set of all states reachable from $x(0) = 0$, and the controllable subspace \mathcal{C} is the set of all $x(0)$ such that $x(T) = 0$ for some $u(t)$ and time $T > 0$. From previous work it is known that

$$\mathcal{R} = \mathcal{V}^* \cap \mathcal{S}_* . \quad (4.5)$$

$$\mathcal{C} = (\mathcal{V}^* + \mathcal{N}(E)) \cap \mathcal{S}_* . \quad (4.6)$$

We show how to select a feedback $u = Kx$ using a minimal polynomial basis for the right nullspace of $R(s)$ that assigns the eigenvalues of the closed-loop system

$$\dot{E}x = (A + BK)x \quad (4.7)$$

while guaranteeing closed-loop regularity. Known results were extended by showing the relation between the possible closed-loop structure and the column-minimal indices of $R(s)$, which we defined as the controllability indices of (2.1).

It was shown that the controllability indices may be generated from the dimensions of the subspaces \mathcal{X}_k and \mathcal{Y}_k in (4.1) and (4.2).

The extension of the notion of the reachability subspace to singular systems is not easy. Consider the feedback

$$u = Kx + Gv \quad (4.8)$$

and define $A_F = (A + BK)$, $B_F = BG$. If there exist F and G such that $(sE - A_F)$ is regular and

$$\mathcal{S} = \mathcal{R}[B_F \quad A_FB_F \quad \dots \quad (A_F)^{n-1}B_F] \quad (4.9)$$

then \mathcal{S} is called a reachability subspace for (2.1). The reachable subspace \mathcal{R} is the largest reachability subspace.

We showed the main result that \mathcal{S} is a reachability subspace if and only if

$$A\mathcal{S} \subset E\mathcal{S} + B, \quad (4.10)$$

that is, \mathcal{S} is an (A, E, B) -invariant subspace, and \mathcal{S} is the limit

of (4.2) with $K = S$. This direct generalization of Wonham's result is attained only after considerable work once the correct notion of a "friend" of S has been defined. A straightforward extension of Wonham's notion results in a "bad friend" which will certainly result in closed-loop irregularity unless stringent and unreasonable conditions hold.

We considered conditions under which (2.1) may be regularized by the feedback (4.8) in the event that (2.2) does not hold. This work results in an extension of the notions of reachability and controllability to definitions that are feedback invariant. In the previous literature, the curious situation had obtained that, although it was desired for the controllability properties of (2.1) to be invariant under feedback of the form (4.8), the definitions themselves were not invariant, since (4.8) can destroy regularity in (2.1). This situation was corrected in [9].

In the context of this discussion, it was natural to consider the relation between proportional (P) feedback (4.8) and proportional-plus-derivative (PD) feedback

$$u = Fx + K\dot{x} + Gv. \quad (4.11)$$

It was discovered that the latter offers virtually no advantage over the former. This is an unfortunate state of affairs, because if PD feedback is allowed, the geometric notions extend in a far easier manner. Our results show clearly the need for a geometric theory based on P feedback, since it is easier to implement than PD feedback.

The essential result of [9] is that (2.1) is regularizable by P or PD feedback (the condition is the same) if and only if

$$EV^* + A + B = R^n. \quad (4.12)$$

It is interesting to compare this to the well known condition for controllability at infinity

$$E + AN(E) + B = R^n. \quad (4.13)$$

V. WALSH FUNCTION ANALYSIS

In this section, we summarize the results of [4,6,7,11].

Suppose that $x(t)$ and $u(t)$ in (2.1) are approximated by

$$\begin{aligned} x(t) &\approx F\phi_r(t) \\ u(t) &\approx G\phi_r(t), \end{aligned} \quad (5.1)$$

where

$$\phi_r(t) = [\phi_0(t) \quad \phi_1(t) \quad . \quad . \quad . \quad \phi_{r-1}(t)]^T \quad (5.2)$$

is a set of r Walsh basis functions, orthogonal on a time interval $[0,1)$. For completeness, it is required that $r = 2^q$ for some integer q . Matrix G is known, and F is an unknown matrix to be determined. Since $\phi_0(t) = 1$, the known initial condition $x(0) = x_0$ may be written as

$$x(0) = [x_0 \quad 0 \quad . \quad . \quad . \quad 0] \phi_r(t) \equiv Q\phi_r(t). \quad (5.3)$$

Proceeding as in [4], we obtain a generalized Lyapunov equation which must be solved for F :

$$AFP - EF = -EQ - BGP. \quad (5.4)$$

The operational matrix for integration P is constructed once r has been selected. Thus, the differential equation (2.1) for $x(t)$ has been transformed to an algebraic equation that must be solved for F .

Note the interesting fact that (5.4) is identical to the top portion of (2.4) with appropriate identification of the variables.

In [4], we discuss (5.4), showing when it has solutions and what this means in terms of the admissible initial subspace of system (2.1). The following results are samples.

Theorem 5.1

Let (E,A) be regular. Suppose α_i is a finite relative eigenvalue of (E,A) and μ_j is an eigenvalue of P . Then the generalized Lyapunov equation (5.4) has a unique solution F for all B, G, Q if and only if $\alpha_i \mu_j \neq 1$ for all i and j . That is, if the finite spectrum of (E,A) does not intersect the spectrum of P^{-1} . ■

Theorem 5.2

Suppose that (E,A) is regular and there exists a unique solution F to (5.4). Let $x(0) \in H_1$, the subspace of admissible initial conditions. Then for, $t > 0$, the solution given by (5.4), (5.1a) satisfies $x(t) \in V^*$, the supremal $(A,E,R(B))$ -invariant subspace of (2.1). ■

In [4] are given also an explicit equation for the solution F to (5.4) when (2.1) is in the Weierstrass form, and a recommended solution procedure for that depends on the simultaneous reduction of E and A to a triangular form using the Moler-Stewart algorithm. This also provides a solution in the state-space case which is far more efficient than the traditional Kronecker product solution.

In [11] we apply Walsh analysis to the bilinear system

$$\dot{x} = Ax + \sum_{k=1}^m D_k x u_k + Bu, \quad (5.5)$$

with $u_k(t)$ the components of the input $u(t)$. There, we show that the equation which must be solved for the Walsh coefficient matrix F of $x(t)$ is

$$F - AFP - \sum_{k=1}^m D_k F P G_k = A Q + B G + \sum_{k=1}^m D_k Q G_k, \quad (5.6)$$

which is a new form of Lyapunov-like equation.

Equation (5.6) may be considered an approximate closed-form solution for (5.5), and so it may be useful in optimal and adaptive control. No other convenient closed-form solution exists.

VI. CHAINED AGGREGATION

Since singular systems and chained aggregation are both useful in the analysis of large-scale systems, we were interested in [13] in extending chained aggregation to singular systems.

Given system (2.1), let $D=0$ and matrix C be of full row rank. We say that there exists an order q aggregation of the pair (E, A) with respect to C (or of the pencil $(sE - A)$ with respect to C) if: 1. there exist matrices $K \in R^{q \times q}$, $F \in R^{q \times q}$, $G \in R^{q \times m}$ such that, for $t \geq 0$, the equation

$$K \dot{v} = Fv + Gu \quad (6.1)$$

has a unique solution for all $v(0)$ and $u(t)$, and 2. there exist matrices P and H such that, when $v(0) = Px(0)$ then

$$y = Hv \quad (6.2)$$

for $t \geq 0$. Note that this definition requires the regularity of the pencil $(sK - F)$. Matrix P may be interpreted as a projection of the semistate $x(t)$ onto a semistate $v(t)$ of reduced dimension q .

If there exists an order p aggregation, we say that $(sE - A)$ is completely aggregable with respect to C . Then, without loss of generality, we may assume that $P = C$ and $H = I$.

Our definition requires equality at $t = 0$, so that all the behavior in $y(t) = Cx(t)$, with $x(t)$ given by (2.1), including possibly impulsive behavior, is preserved in $v(t)$, with $v(t)$ given

by (6.1). (A definition of aggregability could also be made which ignores impulsive behavior.) The next result generalizes the conditions of Aoki to singular systems.

Theorem 6.1

Let the $n \times n$ pencil $(sE-A)$ be regular and $C \in \mathbb{R}^{p \times n}$ have full row rank. Then $(sE-A)$ is completely aggregable with respect to C if and only if there exist a regular $p \times p$ pencil $(sK-F)$ and a matrix $Q \in \mathbb{R}^{p \times m}$ such that

$$(sK-F)C = Q(sE-A). \quad (6.3)$$

Then, the correspondence between (2.1) and (6.1) is given by

$$KC = QE, \quad FC = QA, \quad (6.4)$$

and

$$G = QB, \quad (6.5)$$

with $H = I$ in (6.2). ■

We provided a quotient space analysis of aggregation, proving the following results which are based on Fig. 6.1.

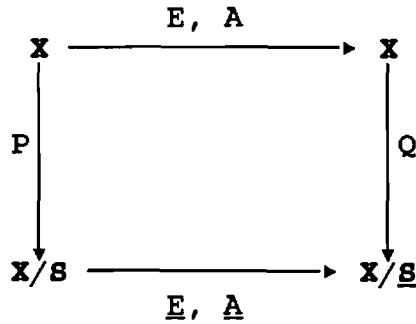


Fig. 6.1

Theorem 6.2

Let $E, A: X \rightarrow X$, with $(sE-A)$ not necessarily regular, and $s \in X$. Define $\underline{s} = Es + As$, and let P, Q be the canonical projections $P: X \rightarrow X/s$ and $Q: X \rightarrow X/\underline{s}$. Then there exist unique maps $\underline{E}, \underline{A}: X/s \rightarrow X/\underline{s}$ such that

$$\underline{E}P = QE \quad (6.6a)$$

$$\underline{A}P = QA. \quad (6.6b)$$

That is, Fig. 6.1 commutes with respect to E, \underline{E} and, separately A, \underline{A} . ■

The conditions for the existence of the induced maps \underline{E} and \underline{A}

in Theorem 6.2 are surprisingly mild; indeed, they always exist! This is a consequence of the definition of \underline{S} , which guarantees that $ES \subset \underline{S}$ and $AS \subset \underline{S}$, which is all the proof requires. The next result shows that if $(sE-A)$ and S satisfy certain additional properties, more can be said about the induced pencil $(sE-A)$.

Theorem 6.3

Let $E, A: X \rightarrow X$, $S \subset X$, and induced maps \underline{E} , \underline{A} be as defined in Theorem 3.2. Suppose $(sE-A)$ is regular. Then $(s\underline{E}-\underline{A})$ is regular if and only if S satisfies

$$\dim(ES + AS) = \dim(S). \quad (6.7)$$

■

A subspace satisfying (6.7) is called a deflating subspace (of (E,A)), and it is spanned by chains or partial chains of (finite and infinite) relative eigenvectors of (E,A) . If $E=I$, then (6.7) reduces to $AS \subset S$, the usual definition of an A -invariant subspace.

The rest of the work in [13] involves notions of observability and the reduction of (2.1) to an upper Hessenberg form like that of Van Dooren. A specialized stable algorithm is given which we call the singular chained aggregation algorithm.

VII. 2-D SINGULAR SYSTEMS

Two results on 2-D singular systems are mentioned here.

A generalization of the Roesser 2-D state model is

$$\begin{bmatrix} E_1 & E_2 \\ E_3 & E_4 \end{bmatrix} \begin{bmatrix} x_{i+1,j}^h \\ x_{i,j+1}^v \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \begin{bmatrix} x_{i,j}^h \\ x_{i,j}^v \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u_{i,j} \quad (7.1)$$

or

$$Ex' = Ax + Bu \quad (7.2)$$

with E possibly singular. The output equation is

$$y = Cx. \quad (7.3)$$

The 2-D Z-transform of (7.2), (7.3) is

$$\begin{aligned} (EZ-A)X &= BU \\ Y &= CX, \end{aligned} \quad (7.4)$$

where

$$Z \equiv \begin{bmatrix} z_1 I & 0 \\ 0 & z_2 I \end{bmatrix}, \quad (7.5)$$

with (z_1, z_2) the 2D Z-transform variables.

By the rank of a polynomial matrix, we mean the normal rank, or the rank over the polynomials.

Theorem 7.1

There exists a solution to (7.2) for every $u_{i,j}$ if and only if

$$\text{rank}[EZ-A \ B] = \text{rank}[EZ-A]. \quad (7.6)$$

The solution, if it exists, is unique with respect to $y_{i,j}$ if and only if

$$\text{rank} \begin{bmatrix} EZ-A \\ -C \end{bmatrix} = \text{rank} [EZ-A] \quad (7.7)$$

■

Now consider the regular case, defined by

$$\det(EZ-A) \neq 0. \quad (7.8)$$

Regularity guarantees that (7.6) and (7.7) hold. By a (2-D) eigenvalue of (E,A) , we mean a complex pair (z_1, z_2) such that (7.8) fails to hold. In the 2-D case, the eigenvalues are generally curves in the plane (z_1, z_2) .

Theorem 7.2

Define $\underline{E} \equiv (EC-A)^{-1}E$, where (c_1, c_2) is any pair such that $\det(EC-A) \neq 0$ and

$$C \equiv \begin{bmatrix} c_1 I & 0 \\ 0 & c_2 I \end{bmatrix}. \quad (7.9)$$

Then $(\alpha_1/\beta_1, \alpha_2/\beta_2)$ is an eigenvalue of (E,A) if and only if $(\beta_1/(c_1\beta_1-\alpha_1), \beta_2/(c_2\beta_2-\alpha_2))$ is an eigenvalue of (I, \underline{E}) . ■

Thus, (E,A) has eigenvalues at infinity if and only if \underline{E} has eigenvalues at zero.

VIII. MISCELLANEOUS RESULTS

Several other results and connections have been derived. They are mentioned on p. 2, and lack of space precludes their discussion

here. copies of this year's publications have been forwarded to NSF.

IX. CONCLUSION

We have given a detailed enough summary of the results achieved under NSF Grant ECS-8518164 to show that the goals proposed in 1986 have been achieved. Nevertheless, our work has also shown that the study of singular systems is a fertile area where much remains to be done.

The study of singular systems yields a curious combination of results which both extend their state-variable analogs and impart a great deal of intuition that was not previously available. This is primarily due to the fascinating interplay between the two matrix operators E and A , which act simultaneously on R^n . The outcome is a richer structure than that obtained by considering only one operator, particularly as far as as the distinction between the domain and codomain spaces goes.

APPENDIX A

SUMMARY OF ACTIVITIES AND RESEARCH

Books and Journal Papers

1. Proceedings of the International Symposium on Singular Systems, Atlanta, GA, Dec. 1987, F.L. Lewis ed., Feb. 1988.
2. F.L. Lewis, "A survey of linear singular systems," Circuits, Systems, and Signal Proc., vol. 5, no. 1, pp. 3-36, 1986.
3. F.L. Lewis, "Further remarks on the Cayley-Hamilton theorem and Leverrier's method for the matrix pencil (sE-A)," IEEE Trans. Automat. Control, Vol. AC-31, No. 9, pp. 869-870, September 1986.
4. F.L. Lewis, B.G. Mertzios, "Analysis of singular systems using orthogonal functions," IEEE Trans. Automat. Control, Vol. AC-32, No. 6, pp. 527-530, June 1987.
5. K. Özçaldıran and F. Lewis, "A geometric approach to eigenstructure assignment for singular systems," IEEE Trans. Automat. Control, Vol. AC-32, No. 7, pp. 629-632, July 1987.
6. F.L. Lewis, M.A. Christodoulou, and B.G. Mertzios, "System inversion using orthogonal functions," Circuits, Systems, Signal Proc., Vol. 6, No. 3, pp. 347-362, 1987.
7. B.G. Mertzios, F.L. Lewis, and G. Vachtsevanos, "On the analysis of singular systems using orthogonal functions," IEE Proc. - Part D, Vol. 135, No. 4, pp. 323-325, July 1988.
8. B.G. Mertzios, M.A. Christodoulou, B.L. Syrmos, and F.L. Lewis, "Direct controllability and observability time-domain conditions of singular systems," IEEE Trans. Automat. Control, Vol. AC-33, No. 8, pp. 788-791, August 1988.
9. K. Özçaldıran and F.L. Lewis, "On the regularizability of singular systems," IEEE Trans. Automat. Control, accepted pending revision, 1988.
10. K. Özçaldıran and F.L. Lewis, "Generalized reachability subspaces for singular systems," SIAM J. Control and Opt., to appear, 1989.
11. F.L. Lewis, V.G. Mertzios, G. Vachtsevanos, and M.A. Christodoulou, "Analysis of bilinear systems using Walsh functions," IEEE Trans. Automat. Control, to appear, 1989.
12. F.L. Lewis and K. Özçaldıran, "Geometric structure and feedback in singular systems," IEEE Trans. Automat. Control, to appear, 1989.

13. F.L. Lewis, M.A. Christodoulou, B.G. Mertzios, and K. Özçaldıran, "Chained aggregation of singular systems," IEEE Trans. Automat. Control, to appear, 1989.
14. B.G. Mertzios and F.L. Lewis, "An algorithm for the computation of the transfer function matrix of generalized 2-D systems," Circuits, Systems, and Signal Proc., to appear 1989.
15. F.L. Lewis and B.G. Mertzios, "On the analysis of discrete linear time-invariant singular systems," IEEE Trans. Automat. Control, to appear, 1989.
16. B.G. Mertzios and F.L. Lewis, "Fundamental matrix of discrete singular systems," Circuits, Systems, and Signal Proc., to appear, 1989.
17. F.L. Lewis, "Geometric design techniques for feedback and observers in singular systems," Automatica, to appear, 1989.
18. F.L. Lewis, "Computational geometry for design in singular systems," IMACS Trans. Scientific Computing, to appear, 1989.

Invited Conference Papers

19. F.L. Lewis and K. Özçaldıran, "Large-scale dynamical interconnections of singular systems," MTNS, Phoenix, AZ, June 1987.
20. F.L. Lewis and G. Beauchamp, "Computation of subspaces for singular systems," MTNS, Phoenix, AZ, June 1987.
21. K. Özçaldıran and F.L. Lewis, "A brief survey of geometric theory for singular systems," ICIAM, Paris, France, June, 1987.
22. F. L. Lewis, "Recent work in singular systems," Proc. Int. Symp. on Singular Systems, pp. 20-24, Atlanta, GA, Dec. 1987.
23. F.L. Lewis, "Subspace recursions and structure algorithms for singular systems," Proc. IEEE Conf. Decision and Control, pp. 1147-1150, Dec. 1987.
24. M.A. Christodoulou, B.G. Mertzios, and F.L. Lewis, "Simplified realization algorithm for singular systems," Proc. IEEE Conf. Decision and Control, pp. 1142-1143, Dec. 1987.
25. F.L. Lewis, "Computational geometry for design in singular systems," Proc. 12-th IMACS World Congress, pp. 184-186, Paris, July 1988.

Other Activities

1. Chairman, Session on "Singular Systems," 25th IEEE Conference on Decision and Control, Athens, Greece, Dec. 1986.
2. Organizer and Joint Chairman, Invited Session on "Singular Systems," 26th IEEE Conference on Decision and Control, Los Angeles, CA, Dec. 1987.
3. Organizer and Chairman, International Symposium on Singular Systems, Atlanta, GA, Dec. 1987.
4. Organizer and Joint Chairman, Invited Session on "Computational Methods for Singular Systems," 12th IMACS World Congress, Paris, July 1988.
5. Guest Editor, Journal of Circuits, Systems, and Signal Processing, Special Issue on Recent Advances in Singular Systems, to appear, 1989.
6. Received a Fulbright Award to perform research in singular systems at Democritus Univ., Xanthi, Greece, Oct.-Nov. 1988.